Off-Shell Bethe Ansatz Equation for osp$(2|1)$ Gaudin Magnets

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Abstract

The semi-classical limit of the algebraic Bethe Ansatz method is used to solve the theory of Gaudin models. Via the off-shell method we find the spectra and eigenvectors of the $N-1$ independent Gaudin Hamiltonians with symmetry $osp(2|1)$. We also show how the off-shell Gaudin equation solves the trigonometric Knizhnik-Zamolodchikov equation.

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1 Introduction

In integrable models of statistical mechanics [1], an important object is the $\mathcal{R}$-matrix $\mathcal{R}(u)$, where u is the spectral parameter. It acts on the tensor product $V^1 \otimes V^2$ for a given vector space V and it is the solution of the Yang-Baxter (YB) equation

$$\mathcal{R}_{12}(u)\mathcal{R}_{13}(u+v)\mathcal{R}_{23}(v) = \mathcal{R}_{23}(v)\mathcal{R}_{13}(u+v)\mathcal{R}_{12}(u) \tag{1.1}$$

in $V^1 \otimes V^2 \otimes V^3$, where $\mathcal{R}_{12} = \mathcal{R} \otimes \mathcal{I}$, $\mathcal{R}_{23} = \mathcal{I} \otimes \mathcal{R}$, etc. and $\mathcal{I}$ is the identity matrix. If $\mathcal{R}$ depends on a Planck-type parameter η so that $\mathcal{R}(u,\eta) = 1 + 2\eta\, r(u) + \mathrm{o}(\eta^2)$, then the "classical r-matrix" obeys the classical YB equation

$$[r_{12}(u), r_{13}(u+v) + r_{23}(v)] + [r_{13}(u+v), r_{23}(v)] = 0. \tag{1.2}$$

Nondegenerate solutions of (1.2) in the tensor product of two copies of simple Lie algebra g , $r_{ij}(u) \in \mathrm{g}_i \otimes \mathrm{g}_j$, $i, j = 1, 2, 3$, were classified by Belavin and Drinfeld [2].

The classical YB equation interplays with conformal field theory in the following way: In the skew-symmetric case $r_{ji}(-u) + r_{ij}(u) = 0$, it is the compatibility condition for the system of linear differential equations

$$\kappa \frac{\partial \Psi(z_1, ..., z_N)}{\partial z_i} = \sum_{j \neq i} r_{ij}(z_i - z_j)\Psi(z_1, ..., z_N) \tag{1.3}$$

in N complex variables $z_1, ..., z_N$ for vector-valued functions Ψ with values in the tensor space $V = V^1 \otimes \cdots \otimes V^N$ and κ is a coupling constant.

In the rational case [2], very simple skew-symmetric solutions are known: $r(u) = \mathrm{C}_2/u$, where $\mathrm{C}_2 \in \mathrm{g} \otimes \mathrm{g}$ is a symmetric invariant tensor of a finite dimensional Lie algebra g acting on a representation space V. Then the corresponding system of linear differential equations (1.3) is the Knizhnik-Zamolodchikov (KZ) equation for the conformal blocks of the Wess-Zumino-Novikov-Witten (WZNW) model of conformal theory on the sphere [3].

The algebraic Bethe Ansatz [4] is the powerful method in the analysis of integrable models. Besides describing the spectra of quantum integrable systems, the Bethe Ansatz also is used to construct exact and manageable expressions for correlation functions [5]. Various representations of correlators in these models were found by Korepin [6], using this method.

Recently, Babujian and Flume [7] developed a method which reveals a link to the algebraic Bethe Ansatz for the theory of the Gaudin model. In their method the wave vectors of the Bethe Ansatz equation for inhomogeneous lattice model render in the semi-classical limit solutions of the KZ equation for the case of simple Lie algebras of

higher rank. More precisely, the algebraic quantum inverse scattering method permits us write the following equation

$$t(u|z)\Phi(u_{1,\cdots,}u_p) = \Lambda(u, u_1, \cdots, u_p|z)\Phi(u_1, \cdots, u_p) - \sum_{\alpha=1}^{p} \frac{\mathcal{F}_\alpha \Phi^\alpha}{u - u_\alpha}. \tag{1.4}$$

Here $t(u|z)$ denotes the transfer matrix of the rational vertex model in an inhomogeneous lattice acting on an N-fold tensor product of $SU(2)$ representation spaces. $\Phi^\alpha = \Phi(u_1, \cdots u_{\alpha-1}, u, u_{\alpha+1}, ..., u_p)$. $\mathcal{F}_\alpha(u_1, \cdots, u_p|z)$ and $\Lambda(u, u_1, \cdots, u_p|z)$ are c numbers. The vanishing of the so-called unwanted terms, $\mathcal{F}_\alpha = 0$, is enforced in the usual procedure of the algebraic Bethe Ansatz by choosing the parameters $u_1, ..., u_p$. In this case the wave vector $\Phi(u_1, \cdots, u_p)$ becomes an eigenvector of the transfer matrix with eigenvalue $\Lambda(u, u_1, \cdots, u_p|z)$. If we keep all unwanted terms, i.e. $\mathcal{F}_\alpha \neq 0$, then the wave vector Φ in general satisfies the equation (1.4), named in [8] as off-shell Bethe Ansatz equation (OSBAE). There is a neat relationship between the wave vector satisfying the OSBAE (1.4) and the vector-valued solutions of the KZ equation (1.3): The general vector valued solution of the KZ equation for an arbitrary simple Lie algebra was found by Schechtman and Varchenko [9]. It can be represented as a multiple contour integral

$$\Psi(z_1, \ldots, z_N) = \oint \cdots \oint \mathcal{X}(u_1, ..., u_p|z)\phi(u_1, ..., u_p|z)du_1 \cdots du_p. \tag{1.5}$$

The complex variables $z_1, ..., z_N$ of (1.5) are related with the disorder parameters of the OSBAE . The vector valued function $\phi(u_1, ..., u_p|z)$ is the semi-classical limit of the wave vector $\Phi(u_1, ..., u_p|z)$. In fact, it is the Bethe wave vector for Gaudin magnets [10], but off mass shell. The scalar function $\mathcal{X}(u_1, ..., u_p|z)$ is constructed from the semi-classical limit of the $\Lambda(u = z_k; u_1, ..., u_p|z)$ and $\mathcal{F}_\alpha(u_1, \cdots, u_p|z)$. This representation of the N-point correlation function shows a deep connection between the inhomogeneous vertex models and the WZNW theory.

In this work we generalize previous results applying the Babujian-Flume ideas for $osp(2|1)$ rational solution of the graded version of the YB equation [11]. It is shown that this ideas persists for the case of semi-classical limit which corresponds to the $osp(2|1)$ trigonometric r-matrix.

The paper is organized as follows. In Section 2 we present the algebraic structure of the $osp(2|1)$ vertex model. Here the inhomogeneous Bethe Ansatz is read from the homogeneous case previously known [12]. We also derive the Off-shell Bethe Ansatz equation for the fundamental representation of the $osp(2|1)$ algebra. In Section 3 , taking into account the semi-classical limit of the results presented in the Section 2, we describe the algebraic structure of the corresponding Gaudin model. In Section 4, data of the off-shell Gaudin equation are used to construct solutions of the trigonometric

KZ equation. In Section 5, our results are extended for the highest representations of the algebra $osp(2|1)$. Conclusions are reserved for Section 6.

2 Structure of the osp(2|1) Vertex Model

We recall that the $osp(2|1)$ algebra is the simplest superalgebra and it can be viewed as the graded version of sl_2. It has three even (bosonic) generators H, $X^{\pm}$ generating a Lie subalgebra sl_2 and two odd (fermionic) generators $V^{\pm}$, whose non-vanishing commutation relations in the Cartan-Weyl basis reads as

$$\begin{aligned} [H, X^{\pm}] &= \pm X^{\pm}, \quad [X^+, X^-] = 2H, \\ [H, V^{\pm}] &= \pm\frac{1}{2}V^{\pm}, \quad [X^{\pm}, V^{\mp}] = V^{\pm}, \quad [X^{\pm}, V^{\pm}] = 0, \\ \{V^{\pm}, V^{\pm}\} &= \pm\frac{1}{2}X^{\pm}, \quad \{V^+, V^-\} = -\frac{1}{2}H. \end{aligned} \tag{2.1}$$

The quadratic Casimir operator is

$$C_2 = H^2 + \frac{1}{2}\{X^+, X^-\} + [V^+, V^-], \tag{2.2}$$

where $\{\cdot\ ,\cdot\}$ denotes the anticommutator and $[\cdot\ ,\cdot]$ the commutator.

The irreducible finite-dimensional representations ρ_j with the highest weight vector are parametrized by half-integer $s = j/2$ or by the integer $j \in N$. Their dimension is $\dim(\rho_j) = 2j + 1$ and the corresponding value of C_2 is $j(j+1)/4 = s(s+1/2)$, $s = 0, 1/2, 1, 3/2, ...$

The representation corresponding to $s = 0$ is the trivial one-dimensional representation. The $s \geq 1/2$ representation contains two isospin multiplets which belong to isospin s and $s - 1/2$, denoted by $|s, s, m\rangle$ and $|s, s-1/2, m\rangle$, respectively. The first quantum number characterizes the representation and the second and third quantum numbers give the isospin and its third component. After a convenient normalization of the states , a given s-representation is defined by

$$\begin{aligned} H\,|s, s, m\rangle &= m\,|s, s, m\rangle\,, \\ X^{\pm}\,|s, s, m\rangle &= \sqrt{(s \mp m)(s \pm m + 1)}\,|s, s, m \pm 1\rangle\,, \\ V^{\pm}\,|s, s, m\rangle &= \pm\frac{1}{2}\sqrt{(s \mp m)}\,|s, s - 1/2, m \pm 1/2\rangle\,, \end{aligned}$$

$$\begin{aligned} H\,|s, s-1/2, m\rangle &= m\,|s, s-1/2, m\rangle\,, \\ X^{\pm}\,|s, s-1/2, m\rangle &= \sqrt{(s - 1/2 \mp m)(s - 1/2 \pm m + 1)}\,|s, s-1/2, m \pm 1\rangle\,, \\ V^{\pm}\,|s, s-1/2, m\rangle &= \pm\frac{1}{2}\sqrt{(s - 1/2 \pm m + 1)}\,|s, s, m \pm 1/2\rangle\,. \end{aligned} \tag{2.3}$$

The fundamental representation has $s = 1/2$ and is given by

$$
\begin{aligned}
H &= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad X^+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\
V^+ &= \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad V^- = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.
\end{aligned}
\tag{2.4}
$$

In (2.4) the basis is $\left|\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\rangle, \left|\frac{1}{2}, 0, 0\right\rangle, \left|\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\rangle$. The first and third vectors will be considered as even and the second as odd, *i.e.*, our grading is BFB.

In the j-representation the odd part has the form [13]:

$$
V^+ = \begin{pmatrix} 0 & V_{j-1} & 0 & \cdots & 0 \\ 0 & 0 & V_{j-2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & V_{-j} \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix}, \quad V^- = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ W_j & 0 & \ddots & \vdots & \vdots \\ 0 & W_{j-1} & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & W_{-j+1} & 0 \end{pmatrix}, \tag{2.5}
$$

where

$$
\begin{aligned}
(V_{j-1}, V_{j-2}, V_{j-3}, ..., V_{-j}) &= \frac{1}{2}(\sqrt{j}, \sqrt{1}, \sqrt{j-1}, \sqrt{2}, ..., \sqrt{1}, \sqrt{j}), \\
(W_j, W_{j-1}, W_{j-2}, ..., W_{-j+1}) &= \frac{1}{2}(-\sqrt{j}, \sqrt{1}, -\sqrt{j-1}, \sqrt{2}, ..., -\sqrt{1}, \sqrt{j}).
\end{aligned}
\tag{2.6}
$$

For the even part we can see from (2.3) that H is diagonal and always has eigenvalue 0 due to isospin integer:

$$
H = \frac{1}{2}\mathrm{diag}(j, j-1, ..., 1, 0, -1, ..., -j). \tag{2.7}
$$

Moreover, $X^\pm$ are given by the sl_2 composition which results in a clear relation with the odd part: $X^\pm = \pm 4(V^\pm)^2$.

2.1 Off-Shell Bethe Ansatz Equation

Consider $V = V_0 \oplus V_1$ a Z_2-graded vector space where 0 and 1 denote the even and odd parts respectively. The components of a linear operator $A \overset{s}{\otimes} B$ in the graded tensor product space $V \overset{s}{\otimes} V$ result in matrix elements of the form

$$
(A \overset{s}{\otimes} B)_{\alpha\beta}^{\gamma\delta} = (-)^{p(\beta)(p(\alpha)+p(\gamma))} \; A_{\alpha\gamma} B_{\beta\delta} \tag{2.8}
$$

and the action of the permutation operator $\mathcal{P}$ on the vector $|\alpha\rangle \overset{s}{\otimes} |\beta\rangle \in V \overset{s}{\otimes} V$ is given by

$$\mathcal{P}\ |\alpha\rangle \overset{s}{\otimes} |\beta\rangle = (-)^{p(\alpha)p(\beta)}\, |\beta\rangle \overset{s}{\otimes} |\alpha\rangle \Longrightarrow (\mathcal{P})^{\gamma\delta}_{\alpha\beta} = (-)^{p(\alpha)p(\beta)}\delta_{\alpha\beta}\ \delta_{\gamma\delta}, \tag{2.9}$$

where $p(\alpha) = 1$ (0) if $|\alpha\rangle$ is an odd (even) element.

Besides $\mathcal{R}$ we have to consider matrices $R = \mathcal{P}\mathcal{R}$ which satisfy

$$R_{12}(u)R_{23}(u+v)R_{12}(v) = R_{23}(v)R_{12}(u+v)R_{23}(u). \tag{2.10}$$

The regular solution of the graded YB equation for the fundamental representation of $osp(2|1)$ algebra was found by Bazhanov and Shadrikov in [14]. It has the form

$$R(u,\eta) = \left(\begin{array}{ccc|ccc|ccc}
x_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & y_5 & 0 & x_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & y_7 & 0 & y_6 & 0 & x_3 & 0 & 0 \\
\\
0 & x_2 & 0 & x_5 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -y_6 & 0 & -x_4 & 0 & -x_6 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & y_5 & 0 & x_2 & 0 \\
\\
0 & 0 & x_3 & 0 & x_6 & 0 & x_7 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & x_2 & 0 & x_5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1
\end{array}\right), \tag{2.11}$$

where

$$\begin{aligned}
x_1 &= \sinh(u+2\eta)\sinh(u+3\eta), \quad x_2 = \sinh u \sinh(u+3\eta), \\
x_3 &= \sinh u \sinh(u+\eta), \quad x_4(u) = \sinh u \sinh(u+3\eta) - \sinh 2\eta \sinh 3\eta, \\
x_5 &= \mathrm{e}^{-u} \sinh 2\eta \sinh(u+3\eta), \qquad y_5 = \mathrm{e}^{u} \sinh 2\eta \sinh(u+3\eta), \\
x_6 &= -\epsilon \mathrm{e}^{-u-2\eta} \sinh 2\eta \sinh u, \qquad y_6 = \epsilon \mathrm{e}^{u+2\eta} \sinh 2\eta \sinh u, \\
x_7 &= \mathrm{e}^{-u} \sinh 2\eta\ \left(\sinh(u+3\eta) + \mathrm{e}^{-\eta}\sinh u\right), \\
y_7 &= \mathrm{e}^{u} \sinh 2\eta\ \left(\sinh(u+3\eta) + \mathrm{e}^{\eta}\sinh u\right).
\end{aligned} \tag{2.12}$$

where $\epsilon = \pm 1$. Here we have assumed that the grading of threefold space is $p(1) = p(3) = 0$ and $p(2) = 1$ and we will choose the solution with $\epsilon = 1$.

Let us consider the inhomogeneous vertex model, where to each vertex we associate two parameters: the global spectral parameter u and the disorder parameter z. In this case, the vertex weight matrix $\mathcal{R}$ depends on $u - z$ and consequently the monodromy matrix will be a function of the disorder parameters z_i.

The graded quantum inverse scattering method is characterized by the monodromy matrix $T(u|z)$ satisfying the equation

$$R(u-v)\left[T(u|z)\stackrel{s}{\otimes}T(v|z)\right]=\left[T(v|z)\stackrel{s}{\otimes}T(u|z)\right]R(u-v), \tag{2.13}$$

whose consistency is guaranteed by the graded version of the YB equation (1.1). $T(u|z)$ is a matrix in the space V (the auxiliary space) whose matrix elements are operators on the states of the quantum system (the quantum space, which will also be the space V). The monodromy operator $T(u|z)$ is defined as an ordered product of local operators $\mathcal{L}_n$ (Lax operator), on all sites of the lattice:

$$T(u|z)=\mathcal{L}_N(u-z_N)\mathcal{L}_{N-1}(u-z_{N-1})\cdots\mathcal{L}_1(u-z_1). \tag{2.14}$$

The Lax operator on the n^{th} quantum space is given the normalized graded permutation of (2.11):

$$\begin{aligned}\mathcal{L}_n &= \frac{1}{x_2}\begin{pmatrix} x_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_2 & 0 & x_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_3 & 0 & x_6 & 0 & x_7 & 0 & 0 \\ & & & & & & & & \\ 0 & y_5 & 0 & x_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & y_6 & 0 & x_4 & 0 & x_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_2 & 0 & x_5 & 0 \\ & & & & & & & & \\ 0 & 0 & y_7 & 0 & y_6 & 0 & x_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & y_5 & 0 & x_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1 \end{pmatrix} \\ &= \begin{pmatrix} L_{11}^{(n)}(u-z_n) & L_{12}^{(n)}(u-z_n) & L_{13}^{(n)}(u-z_n) \\ L_{21}^{(n)}(u-z_n) & L_{22}^{(n)}(u-z_n) & L_{23}^{(n)}(u-z_n) \\ L_{31}^{(n)}(u-z_n) & L_{32}^{(n)}(u-z_n) & L_{33}^{(n)}(u-z_n) \end{pmatrix}\end{aligned} \tag{2.15}$$

Note that $L_{\alpha\beta}^{(n)}(u),\ \alpha,\beta=1,2,3$ are 3 by 3 matrices acting on the n^{th} site of the lattice. It means that the monodromy matrix has the form

$$T(u|z)=\begin{pmatrix} A_1(u|z) & B_1(u|z) & B_2(u|z) \\ C_1(u|z) & A_2(u|z) & B_3(u|z) \\ C_2(u|z) & C_3(u|z) & A_3(u|z) \end{pmatrix}, \tag{2.16}$$

where

$$\begin{aligned} T_{ij}(u|z) &= \sum_{k_1,\dots,k_{N-1}=1}^{3} L_{ik_1}^{(N)}(u-z_N)\stackrel{s}{\otimes}L_{k_1k_2}^{(N-1)}(u-z_{N-1})\stackrel{s}{\otimes}\cdots\stackrel{s}{\otimes}L_{k_{N-1}j}^{(1)}(u-z_1). \\ i,j &= 1,2,3. \end{aligned} \tag{2.17}$$

The vector in the quantum space of the monodromy matrix $T(u|z)$ that is annihilated by the operators $T_{ij}(u|z)$, $i > j$ ($C_i(u|z)$ operators, $i = 1, 2, 3$) and it is also an eigenvector for the operators $T_{ii}(u|z)$ ($A_i(u|z)$ operators, $i = 1, 2, 3$) is called a highest vector of the monodromy matrix $T(u|z)$.

The transfer matrix $\tau(u|z)$ of the corresponding integrable spin model is given by the supertrace of the monodromy matrix in the space V

$$\tau(u|z) = \sum_{i=1}^{3} (-1)^{p(i)} \; T_{ii}(u|z) = A_1(u|z) - A_2(u|z) + A_3(u|z). \tag{2.18}$$

Algebraic Bethe Ansatz solution for the inhomogeneous $osp(2|1)$ vertex model can be obtained from the homogeneous case [12]. The only modification is a local shift of the spectral parameter $u \to u - z_i$.

Here we will define some functions that will be used in the calculations of the Bethe Ansatz:

$$\begin{aligned}
z(u) &= \frac{x_1(u)}{x_2(u)} = \frac{\sinh(u+2\eta)}{\sinh u}, \quad y(u) = \frac{x_3(u)}{y_6(u)} = \frac{\sinh(u+\eta)}{\mathrm{e}^{u+2\eta}\sinh 2\eta}, \\
\omega(u) &= -\frac{x_1(u)x_3(u)}{x_4(u)x_3(u) - x_6(u)y_6(u)} = -\frac{\sinh(u+\eta)}{\sinh(u-\eta)}, \\
\mathcal{Z}(u_k - u_j) &= \begin{cases} z(u_k - u_j) & \text{if} \quad k > j \\ z(u_k - u_j)\omega(u_j - u_k) & \text{if} \quad k < j \end{cases}.
\end{aligned} \tag{2.19}$$

We start defining the highest vector of the monodromy matrix $T(u|z)$ in a lattice of N sites as the even (bosonic) completely unoccupied state

$$|0\rangle = \otimes_{a=1}^{N} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_a . \tag{2.20}$$

Using (2.17) we can compute the normalized action of the monodromy matrix entries on this state

$$\begin{aligned}
A_i(u|z)\,|0\rangle &= X_i(u|z)\,|0\rangle\,, \quad C_i(u|z)\,|0\rangle = 0, \quad B_i(u|z)\,|0\rangle \neq \{0, |0\rangle\}\,, \\
X_i(u|z) &= \prod_{a=1}^{N} \frac{x_i(u-z_a)}{x_2(u-z_a)}, \qquad i = 1, 2, 3.
\end{aligned} \tag{2.21}$$

The Bethe vectors are defined as normal ordered states $\Psi_n(u_1, \cdots, u_n)$ which can be written with aid of a recurrence formula [15]:

$$\Psi_n(u_1, ..., u_n|z) = B_1(u_1|z)\Psi_{n-1}(u_2, ..., u_n|z)$$

$$-B_2(u_1|z)\sum_{j=2}^{n}\frac{X_1(u_j|z)}{y(u_1-u_j)}\prod_{k=2,k\neq j}^{n}\mathcal{Z}(u_k-u_j)\Psi_{n-2}(u_2,...,\hat{u}_j,...,u_n|z)\ , \qquad (2.22)$$

with the initial condition $\Psi_0=|0\rangle\ ,\quad \Psi_1(u_1|z)=B_1(u_1|z)\,|0\rangle$. Here $\hat{u}_j$ denotes that the rapidity u_j is absent: $\Psi(\hat{u}_j\,|z)=\Psi(u_1,...,u_{j-1},u_{j+1},\cdots,u_n|z)$.

The action of the transfer matrix $\tau(u|z)$ on these Bethe vectors gives us the following off-shell Bethe Ansatz equation for the $osp(2|1)$ vertex model

$$\tau(u|z)\Psi_n(u_1,...,u_n|z)=\Lambda_n\Psi_n(u_1,...,u_n|z)-\sum_{j=1}^{n}\mathcal{F}_j^{(n-1)}\Psi_{(n-1)}^{j}+\sum_{j=2}^{n}\sum_{l=1}^{j-1}\mathcal{F}_{lj}^{(n-2)}\Psi_{(n-2)}^{lj}. \qquad (2.23)$$

Let us now describe each term which appear in the right hand side of (2.23) (for more details the reader can see [12]): In the first term the Bethe vectors (2.22) are multiplied by c-numbers $\Lambda_n=\Lambda_n(u,u_1,...,u_n|z)$ given by

$$\Lambda_n=X_1(u|z)\prod_{k=1}^{n}z(u_k-u)-(-)^nX_2(u|z)\prod_{k=1}^{n}\frac{z(u-u_k)}{\omega(u-u_k)}+X_3(u|z)\prod_{k=1}^{n}\frac{x_2(u-u_k)}{x_3(u-u_k)}. \qquad (2.24)$$

The second term is a sum of new vectors

$$\Psi_{(n-1)}^{j}=\left(\frac{x_5(u_j-u)}{x_2(u_j-u)}B_1(u|z)+\frac{1}{y(u-u_j)}B_3(u|z)\right)\Psi_{n-1}(\hat{u}_j), \qquad (2.25)$$

multiplied by scalar functions $\mathcal{F}_j^{(n-1)}$ given by

$$\mathcal{F}_j^{(n-1)}=X_1(u_j|z)\prod_{k\neq j}^{n}\mathcal{Z}(u_k-u_j)+(-)^nX_2(u_j|z)\prod_{k\neq j}^{n}\mathcal{Z}(u_j-u_k). \qquad (2.26)$$

Finally, the last term is a coupled sum of a third type of vector-valued functions

$$\Psi_{(n-2)}^{lj}=B_2(u|z)\Psi_{n-2}(\hat{u}_l,\hat{u}_j), \qquad (2.27)$$

with intricate coefficients

$$\begin{aligned}\mathcal{F}_{lj}^{(n-2)} \;=\;& G_{lj}X_1(u_l|z)X_1(u_j|z)\prod_{k=1,k\neq j,l}^{n}\mathcal{Z}(u_k-u_l)\mathcal{Z}(u_k-u_j)\\ &-(-)^nY_{lj}X_1(u_l|z)X_2(u_j|z)\prod_{k=1,k\neq j,l}^{n}\mathcal{Z}(u_k-u_l)\mathcal{Z}(u_j-u_k)\\ &-(-)^nF_{lj}X_1(u_j|z)X_2(u_l|z)\prod_{k=1,k\neq j,l}^{n}\mathcal{Z}(u_l-u_k)\mathcal{Z}(u_k-u_j)\\ &+H_{lj}X_2(u_l|z)X_2(u_j|z)\prod_{k=1,k\neq j,l}^{n}\mathcal{Z}(u_j-u_k)\mathcal{Z}(u_l-u_k). \qquad (2.28)\end{aligned}$$

where G_{lj} , Y_{lj} , F_{lj} and H_{lj} are additional ratio functions defined by

$$\begin{aligned}
G_{lj} &= \frac{x_7(u_l-u)}{x_3(u_l-u)}\frac{1}{y(u_l-u_j)} + \frac{z(u_l-u)}{\omega(u_l-u)}\frac{x_5(u_j-u)}{x_2(u_j-u)}\frac{1}{y(u-u_l)}, \\
H_{lj} &= \frac{y_7(u-u_l)}{x_3(u-u_l)}\frac{1}{y(u_l-u_j)} - \frac{y_5(u-u_l)}{x_3(u-u_l)}\frac{1}{y(u-u_j)}, \\
Y_{lj} &= \frac{1}{y(u-u_l)}\left\{z(u-u_l)\frac{y_5(u-u_j)}{x_2(u-u_j)} - \frac{y_5(u-u_l)}{x_2(u-u_l)}\frac{y_5(u_l-u_j)}{x_2(u_l-u_j)}\right\}, \\
F_{lj} &= \frac{y_5(u-u_l)}{x_2(u-u_l)}\left\{\frac{y_5(u_l-u_j)}{x_2(u_l-u_j)}\frac{1}{y(u-u_l)} + \frac{z(u-u_l)}{\omega(u-u_l)}\frac{1}{y(u-u_j)}\right. \\
&\quad \left. -\frac{y_5(u-u_l)}{x_2(u-u_l)}\frac{1}{y(u_l-u_j)}\right\}. \qquad (2.29)
\end{aligned}$$

In the usual Bethe Ansatz method, the next step consist in impose the vanishing of the so-called unwanted terms of (2.23) in order to get an eigenvalue problem for the transfer matrix:

We impose $\mathcal{F}_j^{(n-1)} = 0$ and $\mathcal{F}_{lj}^{(n-2)} = 0$ into (2.23) to recover the eigenvalue problem. This means that $\Psi_n(u_1, ..., u_n|z)$ is an eigenstate of $\tau(u|z)$ with eigenvalue Λ_n , provided the rapidities u_j are solutions of the inhomogeneous Bethe Ansatz equations

$$\begin{aligned}
\prod_{a=1}^{N} z(u_j - z_a) &= (-)^{n+1} \prod_{k=1,\ k\neq j}^{n} \frac{z(u_j-u_k)}{z(u_k-u_j)}\omega(u_k-u_j), \\
j &= 1, 2, ..., n. \qquad (2.30)
\end{aligned}$$

3 Structure of the osp$(2|1)$ Gaudin Model

In this section we will consider the theory of the Gaudin model. To do this we need to calculate the semi-classical limit of the results presented in the previous section.

In order to expand the matrix elements of $T(u|z)$, up to an appropriate order in η, we will start by expanding the Lax operator entries defined in (2.15):

$$\begin{aligned}
L_{11}^{(n)} &= \mathcal{I}_n + 2\eta\coth(u-z_n)\ \mathcal{H}_n + 2\eta^2\left(\mathcal{H}_n^2 + \frac{3}{2}\frac{\mathcal{H}_n^2 - \mathcal{H}_n}{\sinh(u-z_n)^2}\right) + \mathrm{o}(\eta^3), \\
L_{22}^{(n)} &= \mathcal{I}_n - 2\eta^2\frac{3\,(\mathcal{I}_n - \mathcal{H}_n^2)}{\sinh(u-z_n)^2} + \mathrm{o}(\eta^3), \\
L_{33}^{(n)} &= \mathcal{I}_n - 2\eta\coth u\ \mathcal{H}_n + 2\eta^2\left(\mathcal{H}_n^2 + \frac{3}{2}\frac{\mathcal{H}_n^2 + \mathcal{H}_n}{\sinh(u-z_n)^2}\right) + \mathrm{o}(\eta^3). \qquad (3.1)
\end{aligned}$$

and for the elements out of the diagonal we have

$$
\begin{aligned}
L_{12}^{(n)} &= -2\eta\,\frac{\mathrm{e}^{-u+z_n}}{\sinh(u-z_n)}\mathcal{V}_n^- + \mathrm{o}(\eta^2), \qquad L_{21}^{(n)} = 2\eta\,\frac{\mathrm{e}^{u-z_n}}{\sinh(u-z_n)}\mathcal{V}_n^+ + \mathrm{o}(\eta^2),\\
L_{23}^{(n)} &= 2\eta\,\frac{\mathrm{e}^{-u+z_n}}{\sinh(u-z_n)}\mathcal{V}_n^- + \mathrm{o}(\eta^2), \qquad L_{32}^{(n)} = 2\eta\,\frac{\mathrm{e}^{u-z_n}}{\sinh(u-z_n)}\mathcal{V}_n^+ + \mathrm{o}(\eta^2),\\
L_{13}^{(n)} &= 2\eta\,\frac{\mathrm{e}^{-u+z_n}}{\sinh(u-z_n)}\mathcal{X}_n^- + \mathrm{o}(\eta^2), \qquad L_{31}^{(n)} = 2\eta\,\frac{\mathrm{e}^{u-z_n}}{\sinh(u-z_n)}\mathcal{X}_n^+ + \mathrm{o}(\eta^2). \qquad (3.2)
\end{aligned}
$$

where $\mathcal{V}^\pm = 2V^\pm$, $\mathcal{X}^\pm = 2X^\pm$ and $\mathcal{H} = 2H$.

Substituting (3.1) and (3.2) into (2.17) we will get the semi-classical expansion for the monodromy matrix entries:

$$
\begin{aligned}
A_1(u|z) &= \mathcal{I} + 2\eta\sum_{a=1}^{N}\coth(u-z_a)\mathcal{H}_a + 4\eta^2\left\{\sum_{a<b}\coth(u-z_a)\coth(u-z_b)\mathcal{H}_a \overset{s}{\otimes} \mathcal{H}_b\right.\\
&\quad + \sum_{a<b}\frac{\mathrm{e}^{z_a-z_b}}{\sinh(u-z_a)\sinh(u-z_b)}\left(\mathcal{X}_a^- \overset{s}{\otimes} \mathcal{X}_b^+ - \mathcal{V}_a^- \overset{s}{\otimes} \mathcal{V}_b^+\right)\\
&\quad \left. + \frac{1}{2}\sum_{a=1}^{N}\left(\mathcal{H}_a^2 + \frac{3}{2}\frac{\mathcal{H}_a^2 - \mathcal{H}_a}{\sinh(u-z_a)^2}\right)\right\} + \mathrm{o}(\eta^3), \qquad (3.3)
\end{aligned}
$$

$$
\begin{aligned}
A_2(u|z) &= \mathcal{I} - 4\eta^2\left\{\sum_{a<b}\frac{\mathrm{e}^{-z_a+z_b}\mathcal{V}_a^+ \overset{s}{\otimes} \mathcal{V}_b^- - \mathrm{e}^{z_a-z_b}\mathcal{V}_a^- \overset{s}{\otimes} \mathcal{V}_b^+}{\sinh(u-z_a)\sinh(u-z_b)} + \frac{3}{2}\sum_{a=1}^{N}\frac{\mathcal{I}_a - \mathcal{H}_a^2}{\sinh(u-z_a)^2}\right\}\\
&\quad + \mathrm{o}(\eta^3), \qquad (3.4)
\end{aligned}
$$

$$
\begin{aligned}
A_3(u|z) &= \mathcal{I} - 2\eta\sum_{a=1}^{N}\coth(u-z_a)\mathcal{H}_a + 4\eta^2\left\{\sum_{a<b}\coth(u-z_a)\coth(u-z_b)\mathcal{H}_a \overset{s}{\otimes} \mathcal{H}_b\right.\\
&\quad + \sum_{a<b}\frac{\mathrm{e}^{-z_a+z_b}}{\sinh(u-z_a)\sinh(u-z_b)}\left(\mathcal{X}_a^+ \overset{s}{\otimes} \mathcal{X}_b^- + \mathcal{V}_a^+ \overset{s}{\otimes} \mathcal{V}_b^-\right)\\
&\quad \left. + \frac{1}{2}\sum_{a=1}^{N}\left(\mathcal{H}_a^2 + \frac{3}{2}\frac{\mathcal{H}_a^2 + \mathcal{H}_a}{\sinh(u-z_a)^2}\right)\right\} + \mathrm{o}(\eta^3). \qquad (3.5)
\end{aligned}
$$

For off-diagonal elements we only need to expand them up to the first order in η

$$
\begin{aligned}
B_1(u|z) &= -B_3(u|z) = -2\eta\sum_{a=1}^{N}\frac{\mathrm{e}^{-u+z_a}}{\sinh(u-z_a)}\mathcal{V}_a^- + \mathrm{o}(\eta^2),\\
C_1(u|z) &= C_3(u|z) = 2\eta\sum_{a=1}^{N}\frac{\mathrm{e}^{u-z_a}}{\sinh(u-z_a)}\mathcal{V}_a^+ + \mathrm{o}(\eta^2),
\end{aligned}
$$

$$\begin{aligned}
B_2(u|z) &= 2\eta\sum_{a=1}^{N}\frac{\mathrm{e}^{-u+z_a}}{\sinh(u-z_a)}\mathcal{X}_a^- + \mathrm{o}(\eta^2),\\
C_2(u|z) &= 2\eta\sum_{a=1}^{N}\frac{\mathrm{e}^{u-z_a}}{\sinh(u-z_a)}\mathcal{X}_a^+ + \mathrm{o}(\eta^2).
\end{aligned}\tag{3.6}$$

Therefore, the semi-classical expansion of the transfer matrix (2.18) has the form

$$\tau(u|z) = \mathcal{I} + 8\eta^2\tau^{(2)}(u|z) + \mathrm{o}(\eta^3),\tag{3.7}$$

where

$$\tau^{(2)}(u|z) = \sum_{a=1}^{N}\frac{\mathcal{G}_a(u)}{\mathrm{e}^{u-z_a}\sinh(u-z_a)} + \sum_{a=1}^{N}\left(\frac{1}{2}\mathcal{H}_a^2 + \frac{3}{4}\frac{\mathcal{I}_a}{\sinh(u-z_a)^2}\right),\tag{3.8}$$

with

$$\begin{aligned}
\mathcal{G}_a(u) &= \sum_{b\neq a}\frac{1}{\sinh(z_a-z_b)}\Big\{\cosh(u-z_a)\cosh(u-z_b)\mathcal{H}_a \overset{s}{\otimes} \mathcal{H}_b\\
&\quad + \frac{1}{2}\left(\mathrm{e}^{-z_a+z_b}\mathcal{X}_a^+ \overset{s}{\otimes} \mathcal{X}_b^- + \mathrm{e}^{z_a-z_b}\mathcal{X}_a^- \overset{s}{\otimes} \mathcal{X}_b^+\right)\\
&\quad + \left(\mathrm{e}^{-z_a+z_b}\mathcal{V}_a^+ \overset{s}{\otimes} \mathcal{V}_b^- - \mathrm{e}^{z_a-z_b}\mathcal{V}_a^- \overset{s}{\otimes} \mathcal{V}_b^+\right)\Big\}.
\end{aligned}\tag{3.9}$$

Here we have used the symmetry

$$\mathcal{G}_{ba}(u,z_a,z_b) = \mathcal{P}\ \mathcal{G}_{ab}(u,z_a,z_b)\ \mathcal{P} = \mathcal{G}_{ab}(u,z_b,z_a),\tag{3.10}$$

and the identity

$$\frac{1}{\sinh(u-z_a)\sinh(u-z_b)} = \frac{1}{\sinh(z_a-z_b)}\left(\frac{\mathrm{e}^{-u+z_a}}{\sinh(u-z_a)} - \frac{\mathrm{e}^{-u+z_b}}{\sinh(u-z_b)}\right).\tag{3.11}$$

The Gaudin Hamiltonians are defined as the residue of $\tau(u|z)$ at the point $u = z_a$. This results in N non-local Hamiltonians

$$\begin{aligned}
G_a &= \sum_{b\neq a}^{N}\frac{1}{\sinh(z_a-z_b)}\Big\{\cosh(z_a-z_b)\mathcal{H}_a \overset{s}{\otimes} \mathcal{H}_b + \frac{1}{2}\Big(\mathrm{e}^{-z_a+z_b}\mathcal{X}_a^+ \overset{s}{\otimes} \mathcal{X}_b^-\\
&\quad + \mathrm{e}^{z_a-z_b}\mathcal{X}_a^- \overset{s}{\otimes} \mathcal{X}_b^+\Big) + \mathrm{e}^{-z_a+z_b}\mathcal{V}_a^+ \overset{s}{\otimes} \mathcal{V}_b^- - \mathrm{e}^{z_a-z_b}\mathcal{V}_a^- \overset{s}{\otimes} \mathcal{V}_b^+\Big\},\\
a &= 1,2,...,N.
\end{aligned}\tag{3.12}$$

satisfying

$$\sum_{a=1}^{N} G_a = 0, \quad \frac{\partial G_a}{\partial z_b} = \frac{\partial G_b}{\partial z_a}, \quad [G_a, G_b] = 0, \qquad \forall a,b.\tag{3.13}$$

In the next section we will use the data of the algebraic Bethe Ansatz for the $osp(2|1)$ vertex model to find the exact spectrum and eigenvectors for each of these $N-1$ independent Hamiltonians.

Before doing this, we would like to consider the semi-classical limit of the fundamental commutation relation (2.13) in order to get the $osp(2|1)$ Gaudin algebra:

The semi-classical expansions of T and R can be written in the following form

$$T(u|z) = 1 + 2\eta l(u|z) + o(\eta^2), \quad R(u) = \mathcal{P}\left[1 + 2\eta r(u) + o(\eta^2)\right]. \tag{3.14}$$

From (3.3–3.6) we can see that the "classical l-operator" has the form

$$l(u|z) = \begin{pmatrix} \mathcal{H}(u|z) & -\mathcal{V}^-(u|z) & \mathcal{X}^-(u|z) \\ \mathcal{V}^+(u|z) & 0 & \mathcal{V}^-(u|z) \\ \mathcal{X}^+(u|z) & \mathcal{V}^+(u|z) & -\mathcal{H}(u|z) \end{pmatrix}, \tag{3.15}$$

where

$$\begin{aligned}
\mathcal{H}(u|z) &= \sum_{a=1}^{N} \coth(u-z_a)\mathcal{H}_a, \\
\mathcal{V}^-(u|z) &= \sum_{a=1}^{N} \frac{\mathrm{e}^{-u+z_a}}{\sinh(u-z_a)}\mathcal{V}_a^-, \qquad \mathcal{V}^+(u|z) = \sum_{a=1}^{N} \frac{\mathrm{e}^{u-z_a}}{\sinh(u-z_a)}\mathcal{V}_a^+, \\
\mathcal{X}^-(u|z) &= \sum_{a=1}^{N} \frac{\mathrm{e}^{-u+z_a}}{\sinh(u-z_a)}\mathcal{X}_a^-, \qquad \mathcal{X}^+(u|z) = \sum_{a=1}^{N} \frac{\mathrm{e}^{u-z_a}}{\sinh(u-z_a)}\mathcal{X}_a^+.
\end{aligned} \tag{3.16}$$

The classical r-matrix has the form

$$\begin{aligned}
r(u) &= \frac{1}{\sinh u}\left\{\cosh u\, \mathcal{H} \overset{s}{\otimes} \mathcal{H} + \frac{1}{2}\left(\mathrm{e}^{-u}\ \mathcal{X}^+ \overset{s}{\otimes} \mathcal{X}^- + \mathrm{e}^{u}\ \mathcal{X}^- \overset{s}{\otimes} \mathcal{X}^+\right)\right. \\
&\quad + \mathrm{e}^{-u}\ (\mathcal{H}\mathcal{V}^+ + \mathcal{V}^+\mathcal{H}) \overset{s}{\otimes} (\mathcal{V}^-\mathcal{H} + \mathcal{H}\mathcal{V}^-) \\
&\quad \left. - \mathrm{e}^{u}\ (\mathcal{V}^-\mathcal{H} + \mathcal{H}\mathcal{V}^-) \overset{s}{\otimes} (\mathcal{H}\mathcal{V}^+ + \mathcal{V}^+\mathcal{H})\right\}.
\end{aligned} \tag{3.17}$$

Here we observe that we are in the fundamental representation of the $osp(2|1)$ algebra where the relation

$$(\mathcal{H}\mathcal{V}^+ + \mathcal{V}^+\mathcal{H})(\mathcal{V}^-\mathcal{H} + \mathcal{H}\mathcal{V}^-) = \mathcal{V}^+\mathcal{V}^- \tag{3.18}$$

holds. Therefore, (3.17) is equivalent to that r-matrix constructed out of the quadratic Casimir in a standard way [16]. Indeed it corresponds to the second regular solution ($\epsilon = -1$) presented in (2.12).

Substituting (3.17) and (3.15) into the fundamental relation (2.13), we have

$$\begin{aligned}
&\mathcal{P}l(u|z) \overset{s}{\otimes} l(v|z) + \mathcal{P}r(u-v)\left[l(u|z) \overset{s}{\otimes} 1 + 1 \overset{s}{\otimes} l(v|z)\right] \\
=\ & l(v|z) \overset{s}{\otimes} l(u|z)\mathcal{P} + \left[l(v|z) \overset{s}{\otimes} 1 + 1 \overset{s}{\otimes} l(u|z)\right]\mathcal{P}r(u-v),
\end{aligned} \tag{3.19}$$

whose consistence is guaranteed by the graded classical YB equation.

From (3.19) we can derive (anti)commutation relations between the matrix elements of $l(u|z)$. They are the defining relations of the $osp(2|1)$ Gaudin algebra :

$$\begin{aligned}
[\mathcal{H}(u|z),\mathcal{H}(v|z)] &= 0,\\
[\mathcal{V}^{\mp}(u|z),\mathcal{X}^{\mp}(v|z)] &= [\mathcal{X}^{\mp}(u|z),\mathcal{X}^{\mp}(v|z)]=0,\\
[\mathcal{V}^{\pm}(u|z),\mathcal{X}^{\mp}(v|z)] &= \frac{2\mathrm{e}^{\pm(u-v)}}{\sinh(u-v)}\left[\mathcal{V}^{\mp}(u|z)-\mathcal{V}^{\mp}(v|z)\right],\\
[\mathcal{X}^{-}(u|z),\mathcal{X}^{+}(v|z)] &= \frac{4\mathrm{e}^{-u+v}}{\sinh(u-v)}\left[\mathcal{H}(u|z)-\mathcal{H}(v|z)\right],\\
\{\mathcal{V}^{-}(u|z),\mathcal{V}^{+}(v|z)\} &= \frac{\mathrm{e}^{-u+v}}{\sinh(u-v)}\left[\mathcal{H}(u|z)-\mathcal{H}(v|z))\right],\\
[\mathcal{H}(u|z),\mathcal{V}^{\mp}(v|z)] &= \pm\frac{1}{\sinh(u-v)}\left[\mathrm{e}^{\pm(u-v)}\mathcal{V}^{\mp}(u|z)-\cosh(u-v)\mathcal{V}^{\mp}(v|z)\right],\\
[\mathcal{H}(u|z),\mathcal{X}^{\mp}(v|z)] &= \pm\frac{2}{\sinh(u-v)}\left[\mathrm{e}^{\pm(u-v)}\mathcal{X}^{\mp}(u|z)-\cosh(u-v)\mathcal{X}^{\mp}(v|z)\right],\\
\{\mathcal{V}^{\mp}(u|z),\mathcal{V}^{\mp}(v|z)\} &= \pm\frac{1}{\sinh(u-v)}\left[\mathrm{e}^{\pm(u-v)}\mathcal{X}^{\mp}(u|z)-\mathrm{e}^{\mp(u-v)}\mathcal{X}^{\mp}(v|z))\right]. \qquad (3.20)
\end{aligned}$$

A direct consequence of these relations is the commutativity of $\tau^{(2)}(u|z)$

$$[\tau^{(2)}(u|z),\tau^{(2)}(v|z)]=0, \qquad \forall u,v \qquad (3.21)$$

from which the commutativity of the Gaudin Hamiltonians G_a follows immediately.

3.1 Off-shell Gaudin Equation

In order to get semi-classical limit of the OSBAE (2.23) we first consider the semi-classical expansions of the Bethe vectors defined in (2.22), (2.25) and (2.27):

$$\begin{aligned}
\Psi_n(u_1,...,u_n|z) &= (-2\eta)^n\Phi_n(u_1,...,u_n|z)+\mathrm{o}(\eta^{n+1}),\\
\Psi^{j}_{(n-1)} &= 2(-2\eta)^{n+1}\frac{\mathrm{e}^{u-u_j}}{\sinh(u-u_j)}\mathcal{V}^{-}(u|z)\Phi_{n-1}(\hat{u}_j\,|z)+\mathrm{o}(\eta^{n+2}),\\
\Psi^{lj}_{(n-2)} &= -(-2\eta)^{n-1}\mathcal{X}^{-}(u|z)\Phi_{n-2}(\hat{u}_l,\hat{u}_j\,|z)+\mathrm{o}(\eta^{n}), \qquad (3.22)
\end{aligned}$$

where

$$\begin{aligned}
\Phi_n(u_1,...,u_n|z) &= \mathcal{V}^{-}(u_1|z)\Phi_{n-1}(u_2,...,u_n|z)\\
&\quad -\mathcal{X}^{-}(u_1|z)\sum_{j=2}^{n}\frac{(-)^j\mathrm{e}^{u_1-u_j}}{\sinh(u_1-u_j)}\Phi_{n-2}(\hat{u}_j\,|z), \qquad (3.23)
\end{aligned}$$

with $\Phi_0 = |0\rangle$ and $\Phi_1(u_1|z) = \mathcal{V}^-(u_1|z)\Phi_0$.

Here we would like make a few comments on the structure of these vector-valued functions. In (3.23) they are written in a normal ordered form. Since we are working with fermionic degree of freedom, the function $\Phi_n(u_1,...,u_n|z)$ is totally antisymmetric.

$$\Phi_n(u_1,...,u_{i-1},u_{i+1},u_i,...,u_n|z) = -\Phi_n(u_1,...,u_{i-1},u_i,u_{i+1},...,u_n|z). \qquad (3.24)$$

To see this one can use the Gaudin algebra (3.20). For instance, in its antisymmetric form the Bethe vector Φ_2 reads as

$$\begin{aligned}\Phi_2(u_1,u_2|z) &= \frac{1}{2}[\mathcal{V}^-(u_1|z)\mathcal{V}^-(u_2|z) - \mathcal{V}^-(u_2|z)\mathcal{V}^-(u_1|z)]\Phi_0 \\ &\quad -\frac{1}{2}[\frac{\mathrm{e}^{u_1-u_2}\mathcal{X}^-(u_1|z)+\mathrm{e}^{-u_1+u_2}\mathcal{X}^-(u_2|z)}{\sinh(u_1-u_2)}]\Phi_0. \qquad (3.25)\end{aligned}$$

Now we will consider the semi-classical expansions of the c-number functions presented in the OSBAE (2.23)

$$\Lambda_n = 1+2(-2\eta)^2\Lambda_n^{(2)}+\mathrm{o}(\eta^3), \qquad \mathcal{F}_j^{(n-1)} = (-2\eta)\ (-)^{j+1}\ f_j^{(n-1)}+\mathrm{o}(\eta^2)$$

$$\mathcal{F}_{lj}^{(n-2)} = 2(-2\eta)^3\frac{(-)^{l+j+1}}{\sinh(u_j-u_l)}\left\{\frac{\mathrm{e}^{u-u_j}}{\sinh(u-u_l)}f_l^{(n-1)}+\frac{\mathrm{e}^{u-u_l}}{\sinh(u-u_j)}f_j^{(n-1)}\right\}+\mathrm{o}(\eta^4), \qquad (3.26)$$

where

$$\begin{aligned}\Lambda_n^{(2)} &= \frac{1}{2}(N+n)+\frac{3}{4}\sum_{a=1}^{N}\frac{1}{\sinh(u-z_a)^2}-\sum_{a=1}^{N}\sum_{j=1}^{n}\coth(u-z_a)\coth(u-u_j) \\ &\quad +\sum_{a<b}^{N}\coth(u-z_a)\coth(u-z_b)+\sum_{j<k}^{n}\coth(u-u_j)\coth(u-u_k), \qquad (3.27)\end{aligned}$$

and

$$f_j^{(n-1)} = -\sum_{a=1}^{N}\coth(u_j-z_a)+\sum_{k\neq j}^{n}\coth(u_j-u_k). \qquad (3.28)$$

Substituting these expressions into the (2.23) and comparing the coefficients of the terms $2(-2\eta)^{n+2}$ we get the first non-trivial consequence for the semi-classical limit of the OSBAE:

$$\tau^{(2)}(u|z)\ \Phi_n(u_1,...,u_n|z) = \Lambda_n^{(2)}\ \Phi_n(u_1,...,u_n|z) - \sum_{j=1}^{n}\frac{(-)^j f_j^{(n-1)}\Theta_{(n-1)}^j}{\mathrm{e}^{u_j-u}\sinh(u_j-u)}. \qquad (3.29)$$

Note that in this limit the contributions from $\Psi^{j}_{(n-1)}$ and $\Psi^{lj}_{(n-2)}$ are combined to give a new vector valued function

$$\Theta^{j}_{(n-1)} = \mathcal{V}^{-}(u|z)\ \Phi_{n-1}(\ \hat{u}_j\ |z) - \mathcal{X}^{-}(u|z) \sum_{k=1,\ k\neq j}^{n} \frac{(-)^{k'}\mathrm{e}^{u_j-u_k}}{\sinh(u_j-u_k)}\ \Phi_{n-2}(\ \hat{u}_j, \hat{u}_k\ |z), \tag{3.30}$$

where $k' = k+1$ for $k<j$ and $k' = k$ for $k>j$. Therefore, our graded OSBAE (3.29) is very similar to that presented by Babujian and Flume for simple Lie algebras [17]. This result could be expected since the superalgebra $osp(2|1)$ has many features which make it very close to the Lie algebra [18].

Finally, we take the residue of (3.29) at the point $u = z_a$ to get the off-shell Gaudin equation:

$$\begin{aligned} G_a\Phi_n(u_1,...,u_n|z) &= g_a\Phi_n(u_1,...,u_n|z) - \sum_{l=1}^{n} \frac{(-)^l f_l^{(n-1)}\phi^{l}_{(n-1)}}{\mathrm{e}^{u_l-z_a}\sinh(u_l-z_a)}, \\ a &= 1,2,...,N \end{aligned} \tag{3.31}$$

where g_a is the residue of $\Lambda_n^{(2)}$

$$g_a = \mathrm{res}_{u=z_a}\Lambda_n^{(2)} = \sum_{b\neq a}^{N}\coth(z_a-z_b) - \sum_{l=1}^{n}\coth(z_a-u_l), \tag{3.32}$$

and $\phi^{l}_{(n-1)}$ is the residue of $\Theta^{l}_{(n-1)}$

$$\phi^{j}_{(n-1)} = \mathrm{res}_{u=z_a}\Theta^{j}_{(n-1)} = \mathcal{V}^{-}_a\Phi_{n-1}(\hat{u}_j\ |z) - \mathcal{X}^{-}_a \sum_{k\neq j}^{n} \frac{(-)^{k'}\mathrm{e}^{u_j-u_k}}{\sinh(u_j-u_k)}\Phi_{n-2}(\hat{u}_k, \hat{u}_j\ |z). \tag{3.33}$$

In this way we are arriving to the main result of this paper. The equation (3.31) permits us solve one of the main problem of the Gaudin model, *i.e.*, the determination of the eigenvalues and eigenvectors of the commuting Hamiltonians G_a (3.9): g_a is the eigenvalue of G_a with eigenfunction Φ_n provided u_l are solutions of the following equations $f_j^{(n-1)} = 0$, *i.e.*:

$$\sum_{k\neq j}^{n}\coth(u_j-u_k) = \sum_{a=1}^{N}\coth(u_j-z_a), \quad j=1,2,...,n. \tag{3.34}$$

Moreover, as we will see in the next section, the off-shell Gaudin equation (3.31) provides solutions for the differential equations known as KZ equations.

4 Knizhnik-Zamolodchickov equation

The KZ differential equation

$$\kappa\frac{\partial\Psi(z)}{\partial z_i}=H_i(z)\Psi(z), \tag{4.1}$$

appeared first as a holonomic system of differential equations on conformal blocks in a WZW model of conformal field theory. Here $\Psi(z)$ is a function with values in the tensor product $V_1\otimes\cdots\otimes V_N$ of representations of a simple Lie algebra, $\kappa=k+g$, where k is the central charge of the model, and g is the dual Coxeter number of the simple Lie algebra.

One of the remarkable properties of the KZ system is that the coefficient functions $H_i(z)$ commute and that the form $\omega=\sum_i H_i(z)dz_i$ is closed [19]:

$$\frac{\partial H_j}{\partial z_i}=\frac{\partial H_i}{\partial z_j},\qquad [H_i,H_j]=0. \tag{4.2}$$

In this section we will identify H_i with the $osp(2|1)$ Gaudin Hamiltonians G_a presented in the previous section and show that the corresponding differential equations (4.1) can be solved via the off-shell Bethe Ansatz method.

Let us now define the vector-valued function $\Psi(z_1,...,z_N)$ through multiple contour integrals of the Bethe vectors (3.23)

$$\Psi(z_1,...,z_N)=\oint\cdots\oint\mathcal{X}(u|z)\Phi_n(u|z)du_1...du_n, \tag{4.3}$$

where $\mathcal{X}\ (u|z)=\mathcal{X}\ (u_1,...,u_n,z_1,...,z_N)$ is a scalar function which in this stage is still undefined.

We assume that $\Psi(z_1,...,z_N)$ is a solution of the equations

$$\kappa\frac{\partial\Psi(z_1,...,z_N)}{\partial z_a}=G_a\Psi(z_1,...,z_N),\quad a=1,2,...,N \tag{4.4}$$

where G_a are the Gaudin Hamiltonians (3.9) and κ is a constant.

Substituting (4.3) into (4.4) we have

$$\kappa\frac{\partial\Psi(z_1,...,z_N)}{\partial z_a}=\oint\left\{\kappa\frac{\partial\mathcal{X}(u|z)}{\partial z_a}\Phi_n(u|z)+\kappa\mathcal{X}(u|z)\frac{\partial\Phi_n(u|z)}{\partial z_a}\right\}du, \tag{4.5}$$

where we are using a compact notation $\oint\{\circ\}\,du=\oint\ldots\oint\{\circ\}\ du_1\cdots du_n$.

Using the Gaudin algebra (3.20) one can derive the following non-trivial identity

$$\frac{\partial\Phi_n}{\partial z_a}=\sum_{l=1}^{n}(-)^l\frac{\partial}{\partial u_l}\left(\frac{\mathrm{e}^{-u_l+z_a}\phi^l_{(n-1)}}{\sinh(u_l-z_a)}\right), \tag{4.6}$$

which allows us write (4.5) in the form

$$\kappa\frac{\partial\Psi}{\partial z_a} = \oint\left\{\kappa\frac{\partial\mathcal{X}(u|z)}{\partial z_a}\Phi_n(u|z) - \sum_{l=1}^{n}(-)^l\kappa\frac{\partial\mathcal{X}(u|z)}{\partial u_l}\left(\frac{\mathrm{e}^{-u_l+z_a}\phi^l_{(n-1)}}{\sinh(u_l-z_a)}\right)\right\}du$$
$$+\kappa\sum_{l=1}^{n}(-)^l\oint\frac{\partial}{\partial u_l}\left(\mathcal{X}(u|z)\frac{\mathrm{e}^{-u_l+z_a}\phi^l_{(n-1)}}{\sinh(u_l-z_a)}\right)du. \tag{4.7}$$

It is evident that the last term of (4.7) is vanishes, because the contours are closed. Moreover, if the scalar function $\mathcal{X}(u|z)$ satisfies the following differential equations

$$\kappa\frac{\partial\mathcal{X}(u|z)}{\partial z_a} = g_a\mathcal{X}(u|z), \qquad \kappa\frac{\partial\mathcal{X}(u|z)}{\partial u_j} = f_j^{(n-1)}\mathcal{X}(u|z), \tag{4.8}$$

we are recovering the off-shell Gaudin equation (3.31) from the first term in (4.7).

Taking into account the definition of the scalar functions $f_j^{(n-1)}$(3.28) and g_a (3.32), we can see that the consistency condition of the system (4.8) is insured by the zero curvature conditions $\partial f_j^{(n-1)}/\partial z_a = \partial g_a/\partial u_j$. Moreover, the solution of (4.8) is easily obtained

$$\mathcal{X}(u|z) = \prod_{a<b}^{N}\sinh(z_a-z_b)^{1/\kappa}\prod_{j<k}^{n}\sinh(u_j-u_k)^{1/\kappa}\prod_{a=1}^{N}\prod_{j=1}^{n}\sinh(z_a-u_j)^{-1/\kappa}. \tag{4.9}$$

This function determines the monodromy of $\Psi(z_1,...,z_N)$ as solution of the trigonometric KZ equation (4.4) and these results are in agreement with the Schechtman-Varchenko construction for multiple contour integral as solutions of the KZ equation in an arbitrary simple Lie algebra [9].

5 Highest Representations

The generalization of our results for the highest representations of the $osp(2|1)$ algebra requests the knowledge of the corresponding algebraic Bethe Ansatz which. It can be obtained from the Bethe Ansatz for the fundamental representation using a fusion procedure. Nevertheless, we can use the fact that the one-parameter operator families (3.16) form the highest weight module of the infinite-dimensional Lie superalgebra. As in the sl_2 case presented by Sklyanin in [20], it is characterized by the vacuum $|0\rangle$

$$\mathcal{H}(u|z)\,|0\rangle = h(u|z)\,|0\rangle\,, \qquad \mathcal{V}^{+}(u|z)\,|0\rangle = 0, \qquad \mathcal{X}^{+}(u|z)\,|0\rangle = 0, \tag{5.1}$$

the dual vacuum $\langle 0|$

$$\langle 0|\,\mathcal{H}(u|z) = h(u|z)\,\langle 0|\,, \qquad \langle 0|\,\mathcal{V}^{-}(u|z) = 0, \qquad \langle 0|\,\mathcal{X}^{-}(u|z) = 0, \qquad \langle 0\mid 0\rangle = 1, \tag{5.2}$$

and the highest weight scalar function

$$h(u|z) = \sum_{a=1}^{N} 2s_a \coth(u - z_a), \qquad s_a = \frac{1}{2}, 1, \frac{3}{2}, \cdots \tag{5.3}$$

The solution of the Gaudin eigenvalue problem is given in term of Bethe vectors $\Phi_n(u_1, ..., u_n)$ given by (3.23) with the operators written in the j-representation (2.5) and (2.7).

It is convenient to express the Gaudin Hamiltonians G_a in terms of a generating function $t(u|z)$ which is obtained as the result of replacing H, $X^{\pm}$ and $V^{\pm}$ in the quadratic Casimir (2.2) by $\mathcal{H}(u|z)$, $\mathcal{X}^{\pm}(u|z)$ and $\mathcal{V}^{\pm}(u|z)$, respectively:

$$4t(u|z) = \sum_{a=1}^{N} \frac{\mathcal{G}_a(u)}{\mathrm{e}^{u-z_a}\sinh(u - z_a)} + \sum_{a=1}^{N} \left(\mathcal{H}_a^2 + \frac{s_a(s_a + 1/2)}{\sinh(u - z_a)^2} \right). \tag{5.4}$$

where $\mathcal{G}_a(u)$ is given by (3.9).

Using a theorem [21, 22] which states that $\Phi_n(u_1, ..., u_n)$ is an eigenvector of the commuting operators $\mathcal{G}_a(u)$ or, equivalently, of $t(u|z)$ if and only if the parameters $u_1, ..., u_n$ satisfy the Bethe equations

$$\sum_{a=1}^{N} 2s_a \coth(u - z_a) = \sum_{k=1}^{n} \coth(u - u_k). \tag{5.5}$$

The corresponding eigenvalue $\Lambda(u)$ of $4t(u|z)$ is then

$$\Lambda(u) = \lambda^2(u) - 2\, \partial_u \lambda(u) \tag{5.6}$$

where

$$\lambda(u) = \sum_{a=1}^{N} 2s_a \coth(u - z_a) - \sum_{k=1}^{n} \coth(u - u_k) \tag{5.7}$$

From these results we conjecture that the off-shell Gaudin equation for the highest representations is also given by (3.31) with

$$f_j^{(n-1)} = -\sum_{a=1}^{N} 2s_a \coth(u_j - z_a) + \sum_{k \neq j}^{n} \coth(u_j - u_k), \tag{5.8}$$

and

$$g_a = \sum_{b \neq a} 4s_a s_b \coth(z_a - z_b) - \sum_{l=1}^{n} 2s_a \coth(z_a - u_l). \tag{5.9}$$

Consequently, the monodromy of the function $\Psi(z_1, ..., z_N)$ (4.3) for the j-representation is given by

$$\mathcal{X}(u|z) = \prod_{a<b}^{N} \sinh(z_a - z_b)^{4s_a s_b/\kappa} \prod_{j<k}^{n} \sinh(u_j - u_k)^{1/\kappa} \prod_{a=1}^{N} \prod_{j=1}^{n} \sinh(z_a - u_j)^{-2s_a/\kappa}, \tag{5.10}$$

which also is in agreement with the Schechtman-Varchenko construction [9].

6 Conclusion

In this paper a graded 19-vertex model was used to generalise previous rational results connecting Gaudin magnet models and semi-classical off-shell Bethe Ansatz of vertex models.

Using the semi-classical limit of the transfer matrix of the vertex model we derive the trigonometric $osp(2|1)$ Gaudin Hamiltonians. The reduction of the off-shell Gaudin equation to an eigenvalue equation gives us the exact spectra and eigenvectors for these Gaudin magnets. Data of the off-shell Gaudin equation were used to show that a Jackson-type integral (4.3) is solution of the trigonometric KZ differential equation.

In fact, this method had already been used with success to constructing solutions of trigonometric KZ equations [23, 24] and elliptic KZ-Bernard equations [25], for the six-vertex model and eight-vertex model, respectively.

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